

MTH250 - CALCULUS

Lecture No. 25

DR. ABDUL WAHAB

Assistant Professor,
Department of Mathematics, CIIT, Pakistan.

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COMSATS Institute of Information Technology
CIIT, Virtual Campus Secretariat 402, Street 34, I-8/2 Islamabad-Pakistan.
Ph:+92 51 9101237-39 | info@vcomsats.edu.pk | vcomsats.edu.pk

These lecture notes are prepared to serve as handouts for the video lectures on the course *MTH-250 Calculus* of Virtual Campus, COMSATS Institute of Information Technology, Islamabad, Pakistan. The notes were derived from an extensive collection of notes and books. Most of the theoretical details as well as worked problems are from *Calculus*, by *H. Anton, I Bivens and S. Davis*, John Wiley & Sons, 10th edition, 2012.

Contents

Contents	i
25 Double integrals	1
25.1 Introduction to double integrals	1
25.1.1 Iterated integrals	3
25.2 Integrals over non-rectangular regions	6
25.3 Reversing the order of integration	9

LECTURE 25

Double integrals

25.1 Introduction to double integrals

Recall that the definite integral of a one variable function f was defined as

$$\int_a^b f(x) ds = \lim_{\max \Delta x_k \rightarrow 0} \sum_{k=1}^n f(x_k^*) \Delta x_k = \lim_{n \rightarrow \infty} \sum_{k=1}^n f(x_k^*) \Delta x_k$$

arose from a problem of finding area under curve.

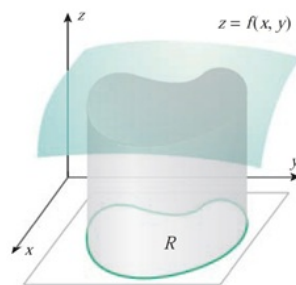


Figure 25.1: Volume enclosed between the surface $z = f(x, y)$ and the region R .

Definition 25.1 (The volume problem).

Given a function f of two variables that is continuous non-negative on a region R in

the xy -plane, find the volume of the solid enclosed between the surface $z = f(x, y)$ and the region R .

In order to address the volume problem, we proceed as follows.

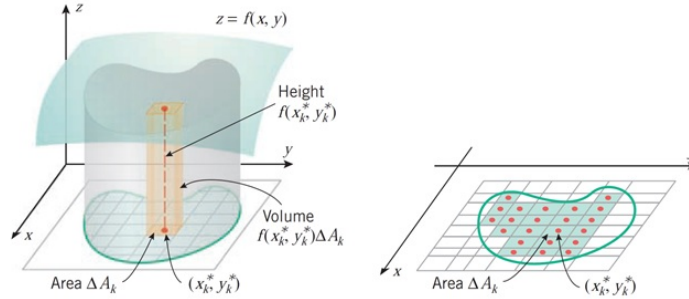


Figure 25.2: Dividing the rectangle enclosing the region R into n subrectangles.

1. Using lines parallel to the coordinate axes, divide the rectangle enclosing the region R into n sub-rectangles.
2. Choose any arbitrary point (x_k^*, y_k^*) in each sub-rectangle.
3. Let ΔA_k denote the area of the k^{th} rectangle.
4. The volume of a rectangular parallelepiped with height $f(x_k^*, y_k^*)$ and base area ΔA_k is given by $f(x_k^*, y_k^*)\Delta A_k$.
5. Approximation to the volume V of the entire solid is

$$R = \sum_{k=1}^n f(x_k^*, y_k^*)\Delta A_k.$$

The sum R is called the n^{th} Riemann sum.

Definition 25.2 (Volume under a surface).

If f is a function of two variables that is continuous and non-negative on a region R in the xy -plane, then the volume of the solid enclosed between the surface $z = f(x, y)$ and the region R is defined by

$$V = \lim_{n \rightarrow +\infty} \sum_{k=1}^n f(x_k^*, y_k^*)\Delta A_k$$

Here $n \rightarrow +\infty$ indicates the process of increasing the number of sub-rectangles of the rectangle enclosing R in such a way that both the lengths and the widths of the sub-rectangles approach zero.

Definition 25.3 (Double integral).

The double integral of a function $f(x, y)$ over a region R is defined as the limit of the Riemann sums and is denoted by

$$\iint_R f(x, y) dA = \lim_{n \rightarrow +\infty} \sum_{k=1}^n f(x_k^*, y_k^*) \Delta A_k.$$

If f is continuous and non-negative on the region R , then the volume V can be related to the double integral as

$$V = \iint_R f(x, y) dA.$$

Definition 25.4 (Partial integration).

The partial derivatives of a function $f(x, y)$ are calculated by holding one of the variables fixed and differentiating with respect to the other variable. If we reverse this process, we call it partial integration. The symbols

$$\int_a^b f(x, y) dx \quad \text{and} \quad \int_c^d f(x, y) dy$$

denote the partial definite integrals with respect to x and y , respectively.

Example 25.1.

$$\int_0^1 xy^2 dx = y^2 \int_0^1 x dx = \frac{y^2 x^2}{2} \Big|_0^1 = \frac{y^2}{2}$$

and

$$\int_0^1 xy^2 dy = x \int_0^1 y^2 dy = \frac{xy^3}{3} \Big|_0^1 = \frac{x}{3}.$$

25.1.1 Iterated integrals

Definition 25.5 (Iterated integrals). A partial definite integral with respect to x is a function of y and hence can be integrated with respect to y ; similarly, a partial definite integral with respect to y can be integrated with respect to x . This two-stage integration process is called iterated (or repeated) integration. We introduce the following notation

$$\int_c^d \int_a^b f(x, y) dx dy = \int_c^d \left[\int_a^b f(x, y) dx \right] dy,$$

$$\int_a^b \int_c^d f(x, y) dy dx = \int_a^b \left[\int_c^d f(x, y) dy \right] dx.$$

These integrals are called iterated integrals.

Example 25.2. Evaluate $\int_1^3 \int_2^4 (40 - 2xy) dy dx$ and $\int_2^4 \int_1^3 (40 - 2xy) dx dy$.

Solution.

$$\begin{aligned}\int_1^3 \int_2^4 (40 - 2xy) dy dx &= \int_1^3 \left[\int_2^4 (40 - 2xy) dy \right] dx = \int_1^3 (40y - xy^2) \Big|_{y=2}^4 dx, \\ &= \int_1^3 [(160 - 16x) - (80 - 4x)] dx = \int_1^3 (80 - 12x) dx \\ &= (80x - 6x^2) \Big|_1^3 = 112\end{aligned}$$

and

$$\begin{aligned}\int_2^4 \int_1^3 (40 - 2xy) dx dy &= \int_2^4 \left[\int_1^3 (40 - 2xy) dx \right] dy = \int_2^4 (40x - x^2y) \Big|_{x=1}^3 dy \\ &= \int_2^4 [(120 - 9y) - (40 - y)] dy = \int_2^4 (80 - 8y) dy \\ &= (80y - 4y^2) \Big|_2^4 = 112.\end{aligned}$$

Therefore,

$$V = \int_1^3 \int_2^4 (40 - 2xy) dy dx = \int_2^4 \int_1^3 (40 - 2xy) dx dy = 112$$

□

Theorem 25.6 (Fubini).

Let R be the rectangle defined by the inequalities

$$a \leq x \leq b, \quad c \leq y \leq d.$$

If $f(x, y)$ is continuous on this rectangle, then

$$\iint_R f(x, y) dA = \int_c^d \int_a^b f(x, y) dx dy = \int_a^b \int_c^d f(x, y) dy dx.$$

Example 25.3. Evaluate the double integral $\iint_R y^2 x dA$ over the rectangle

$$R = \{(x, y) : -3 \leq x \leq 2, \quad 0 \leq y \leq 1\}.$$

Solution.

$$\begin{aligned}\iint_R y^2 x dA &= \int_0^1 \int_{-3}^2 y^2 x dx dy = \int_0^1 \left[\frac{1}{2} y^2 x^2 \right]_{x=-3}^2 dy \\ &= \int_0^1 \left(-\frac{5}{2} y^2 \right) dy = -\frac{5}{6} y^3 \Big|_0^1 = -\frac{5}{6}.\end{aligned}$$

□

Example 25.4. Use a double integral to find the volume of the solid that is bounded above by the plane $z = 4 - x - y$ and below by the rectangle $R = [0, 1] \times [0, 2]$.

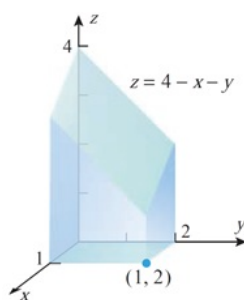


Figure 25.3: Volume of the solid bounded above by the plane $z = 4 - x - y$ and below the rectangle $R = [0, 1] \times [0, 2]$.

Solution.

The volume is the double integral of $z = 4 - x - y$ over R . Using theorem of Fubini, this can be obtained from either of the iterated integrals

$$\int_0^2 \int_0^1 (4 - x - y) dx dy \quad \text{or} \quad \int_0^1 \int_0^2 (4 - x - y) dy dx.$$

Using the first of these, we obtain

$$\begin{aligned} V &= \iint_R (4 - x - y) dA = \int_0^2 \int_0^1 (4 - x - y) dx dy \\ &= \int_0^2 \left[4x - \frac{x^2}{2} - xy \right]_{x=0}^1 dy = \int_0^2 \left(\frac{7}{2} - y \right) dy \\ &= \left[\frac{7}{2}y - \frac{1}{2}y^2 \right]_0^2 = 5 \end{aligned}$$

□

Theorem 25.7 (Properties of double integrals).

1. $\iint_R c f(x, y) dA = c \iint_R f(x, y) dA$ where c is a constant.
2. $\iint_R [f(x, y) + g(x, y)] dA = \iint_R f(x, y) dA + \iint_R g(x, y) dA.$
3. $\iint_R [f(x, y) - g(x, y)] dA = \iint_R f(x, y) dA - \iint_R g(x, y) dA.$
4. $\iint_R f(x, y) dA = \iint_{R_1} f(x, y) dA + \iint_{R_2} f(x, y) dA.$

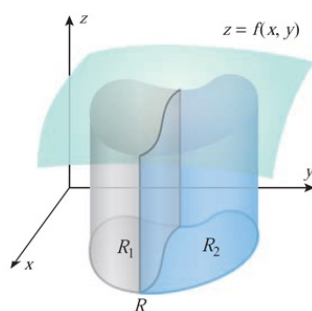


Figure 25.4: The volume of the entire solid is the sum of the volumes of the solids R_1 and R_2 .

25.2 Integrals over non-rectangular regions

In this section, we will consider the iterated integrals over non-rectangular regions, precisely we consider the integrals with variables limits of the form

$$\int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y) dy dx = \int_a^b \left[\int_{g_1(x)}^{g_2(x)} f(x, y) dy \right] dx$$

$$\int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y) dx dy = \int_c^d \left[\int_{h_1(y)}^{h_2(y)} f(x, y) dx \right] dy$$

Definition 25.8. A type I region is bounded on the left and right by vertical lines $x = a$ and $x = b$ and is bounded below and above by continuous curves $y = g_1(x)$ and $y = g_2(x)$, where $g_1(x) \leq g_2(x)$ for $a \leq x \leq b$, see Figure 25.5.

A type II region is bounded below and above by horizontal lines $y = c$ and $y = d$ and is bounded on the left and right by continuous curves $x = h_1(y)$ and $x = h_2(y)$ satisfying $h_1(y) \leq h_2(y)$ for $c \leq y \leq d$, see Figure 25.5.

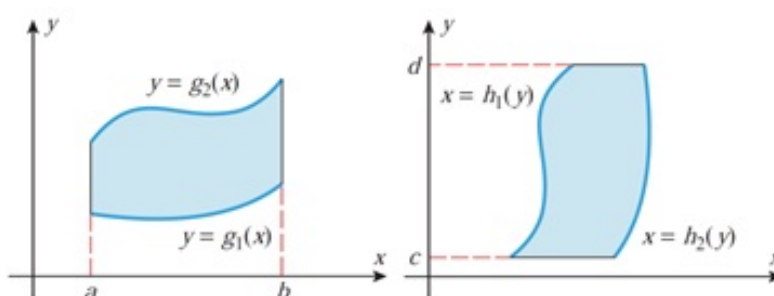


Figure 25.5: Left: type one region. Right: type two region.

Example 25.5. Evaluate $\int_0^1 \int_{-x}^{x^2} y^2 x dy dx$.

Solution.

$$\begin{aligned}\int_0^1 \int_{-x}^{x^2} y^2 x dy dx &= \int_0^1 \left[\int_{-x}^{x^2} y^2 x dy \right] dx = \int_0^1 \frac{y^3 x}{3} \Big|_{y=-x}^{y=x^2} dx \\ &= \int_0^1 \left[\frac{x^7}{3} + \frac{x^4}{3} \right] dx = \left(\frac{x^8}{24} + \frac{x^5}{15} \right) \Big|_0^1 = \frac{13}{120}.\end{aligned}$$

□

Example 25.6. Evaluate $\int_0^{\pi/3} \int_0^{\cos y} x \sin y dx dy$.

Solution.

$$\begin{aligned}\int_0^{\pi/3} \int_0^{\cos y} x \sin y dx dy &= \int_0^{\pi/3} \left[\int_0^{\cos y} x \sin y dx \right] dy = \int_0^{\pi/3} \frac{x^2}{2} \sin y \Big|_{x=0}^{x=\cos y} dy \\ &= \int_0^{\pi/3} \left[\frac{1}{2} \cos^2 y \sin y \right] dy = - \left(\frac{1}{6} \cos^3 y \right) \Big|_0^{\pi/3} = \frac{7}{48}.\end{aligned}$$

□

Theorem 25.9.

If R is a type I region on which $f(x, y)$ is continuous, then

$$\iint_R f(x, y) dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y) dy dx.$$

If R is a type II region on which $f(x, y)$ is continuous, then

$$\iint_R f(x, y) dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y) dx dy.$$

Example 25.7. Evaluate $\iint_R xy dA$ over the region R enclosed between

$$y = \frac{1}{2}x, \quad y = \sqrt{x}, \quad x = 2, \quad x = 4.$$

Solution. We view R as a type I region. The region R and a vertical line corresponding to a fixed x are shown in Figure 25.6. This line meets the region R at the lower boundary $y = \frac{1}{2}x$ and the upper boundary $y = \sqrt{x}$. These are the y -limits of integration. Moving this line first left and then right yields the x -limits of integration, $x = 2$ and $x = 4$. Thus,

$$\begin{aligned}\iint_R xy dA &= \int_2^4 \int_{x/2}^{\sqrt{x}} xy dy dx = \int_2^4 \left[\frac{xy^2}{2} \right]_{y=x/2}^{y=\sqrt{x}} dx \\ &= \int_2^4 \left(\frac{x^2}{2} - \frac{x^3}{8} \right) dx = \left[\frac{x^3}{6} - \frac{x^4}{32} \right]_2^4 \\ &= \left(\frac{64}{6} - \frac{256}{32} \right) - \left(\frac{8}{6} - \frac{16}{32} \right) = \frac{11}{6}.\end{aligned}$$

□

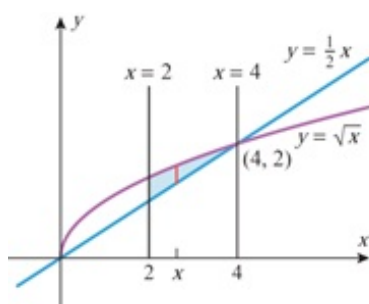


Figure 25.6: R enclosed between $y = \frac{1}{2}x$, $y = \sqrt{x}$, $x = 2$, $x = 4$.

Example 25.8. Use a double integral to find the volume of the tetrahedron bounded by the coordinate planes and the plane $z = 4 - 4x - 2y$.

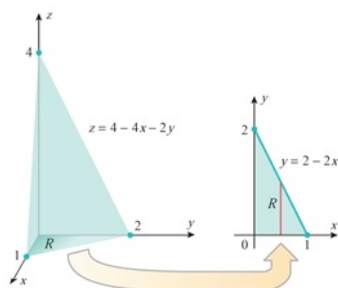


Figure 25.7: The tetrahedron bounded by the coordinate planes and the plane $z = 4 - 4x - 2y$.

Solution. The tetrahedron in question is bounded above by the plane $z = 4 - 4x - 2y$ and below by the triangular region R . Thus, the volume is given by

$$V = \iint_R (4 - 4x - 2y) dA.$$

The region R is bounded by the x -axis, and the line $y = 2 - 2x$ (set $z = 0$) so that treating R as a type I region yields

$$\begin{aligned} V &= \iint_R (4 - 4x - 2y) dA = \int_0^1 \int_0^{2-2x} (4 - 4x - 2y) dy dx \\ &= \int_0^1 [4y - 4xy - y^2]_{y=0}^{2-2x} dx \\ &= \int_0^1 (4 - 8x + 4x^2) dx = \frac{4}{3}. \end{aligned}$$

□

Example 25.9. Find the volume of the solid bounded by $x^2 + y^2 = 4$, $y + z = 4$ and $z = 0$.

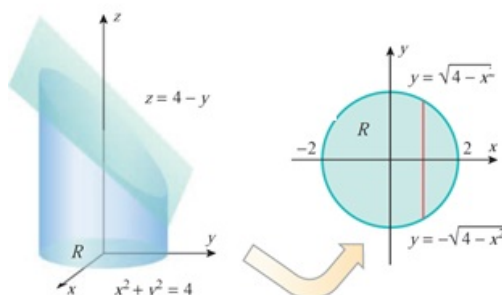


Figure 25.8: The solid bounded by $x^2 + y^2 = 4$, $y + z = 4$ and $z = 0$.

Solution. The solid is bounded by the plane $z = 4 - y$ and below by the region R within the circle $x^2 + y^2 = 4$. The volume is given by

$$V = \iint_R (4 - y) dA.$$

Treating R as a type I region, we obtain

$$\begin{aligned} V &= \int_{-2}^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (4 - y) dy dx \\ &= \int_{-2}^2 \left[\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (4 - y) dy \right] dx \\ &= \int_{-2}^2 8\sqrt{4 - x^2} dx = 8(2\pi) = 16\pi. \end{aligned}$$

□

25.3 Reversing the order of integration

Sometimes the evaluation of an iterated integral can be simplified by reversing the order of integration. The next example illustrates how this is done.

Example 25.10.

Since there is no elementary anti-derivative of e^{x^2} , the integral

$$\int_0^2 \int_{y/2}^1 e^{x^2} dx dy$$

cannot be evaluated by performing the x -integration first. We evaluate this integral by expressing it as an equivalent iterated integral with the order of integration reversed.

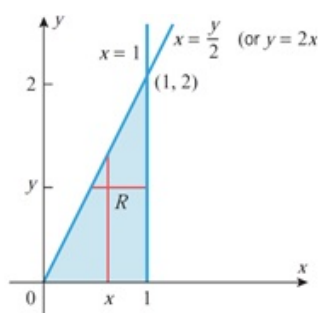


Figure 25.9: The triangular region R of integration.

For the inside integration, y is fixed and x varies from the line $x = \frac{y}{2}$ to the line $x = 1$; see Figure 25.9. For the outside integration, y varies from 0 to 2, so the given iterated integral is equal to a double integral over the triangular region R in Figure 25.9. We treat R as a type I region, which enables us to write the given integral as

$$\begin{aligned} \int_0^2 \int_{y/2}^1 e^{x^2} dx dy &= \iint e^{x^2} dA = \int_0^1 \int_0^{2x} e^{x^2} dy dx \\ &= \int_0^1 \left[e^{x^2} y \right]_{y=0}^{y=2x} dx = \int_0^1 2x e^{x^2} dx = e^{x^2} \Big|_0^1 = e - 1. \end{aligned}$$